Research Statement R. Sean Bowman

The central problem in the theory of 3–dimensional topology is to understand and classify 3–dimensional spaces. Cutting spaces into simple pieces along surfaces, understanding the pieces, and then understanding how they're put together is one way to study a given 3–dimensional space. Heegaard decompositions, which cut a manifold into two handlebodies, were the first such decompositions studied. Max Dehn introduced another decomposition, called Dehn surgery, which cuts a manifold into a solid torus and a knot exterior. Studying which manifolds can be obtained by surgery on which knots has historically been the source of many powerful techniques in low dimensional topology. This is partly due to the subject's close connections with hyperbolic geometry, explored by Thurston.

My research focuses on the bridge numbers of knots, invariants that are closely connected with the geometry and topology of spaces associated with the knot. Bridge numbers are difficult to calculate, and until recently not much was known about the bridge numbers of specific classes of knots. However, the concept of bridge number is central to many open problems in 3–manifold topology. For example, the Berge conjecture, which states that the list of knots with surgeries yielding lens spaces in [3] is complete, can be rephrased by saying that the only knots in lens spaces with a surgery yielding S^3 are 1–bridge.

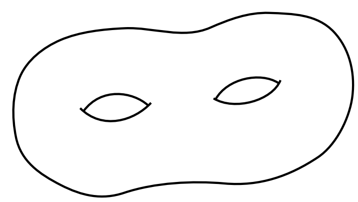
My thesis work involved knots in handlebodies, which I describe in section 1. This work is a continuation of work done in the 1980s and 1990s by Berge, Gabai, and Wu, and it leads to many new and interesting examples of knots. Further study of this class of knots may reveal new examples, techniques, and philosophies in the same way that Berge's knots in S^3 did.

More recently I have been interested in bridge numbers of knots in manifolds without boundary, and I describe this work in section 2. Here the problem is to say meaningful things about the bridge numbers of certain families of knots and to compare the bridge numbers to Dehn surgeries on the knot. For example, Thurston has shown that when the complement of a knot in S^3 admits a hyperbolic structure, all but finitely many surgeries on the knot also admit a hyperbolic structure. We are particularly interested in the cases when this does not occur, and call such surgeries **exceptional**. Ongoing work addresses questions such as what one can say about the bridge numbers or

Heegaard splittings of knots with certain exceptional surgeries.

In addition, I am interested in topological data analysis and more generally in the application of ideas from topology and geometry to real world problems. I describe some of this work in section 3.

1 Bridge number in manifolds with boundary



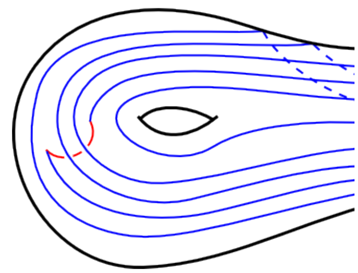
A 3–manifold is a space which is locally homeomorphic to $\mathbb{R}^3$; locally, it looks like the space we inhabit. A handlebody is a special type of 3–manifold obtained from a ball by attaching "handles." The boundary of a handlebody is a surface of genus g, and we also call g the genus of the handlebody. A handlebody of genus 2 is pictured at right. Handlebodies play the role of the simple pieces into which a manifold is cut in the theory of Heegaard splittings [10].

Dehn surgery on a knot K at slope α consists of removing a neighborhood of K from the ambient 3–manifold and gluing back in a solid torus so that the curve α bounds a disk in the resulting manifold. Here α, the surgery slope, is a curve in the boundary of a neighborhood of K. Dehn surgery is fundamental in 3–manifold topology because every 3–manifold can be obtained by Dehn surgery along a link in S^3 [12].

Given the importance of Dehn surgery and handlebody decompositions of manifolds, it's a natural question to ask when a knot in a handlebody has a surgery again yielding a handlebody. Call a knot in a handlebody H which has a nontrivial handlebody surgery an H–knot. A manifold M is said to be ∂–reducible if it contains a properly embedded disk whose boundary is nontrivial in ∂M. Therefore, H–knots have boundary reducible surgeries.

Gabai [9] and Berge [4] both investigated the case of knots in solid tori (handlebodies of genus $g = 1$). Gabai completely classified surgeries on knots in solid tori and also gave an algorithm to determine when such a knot has a handlebody surgery. Berge gave a complete list of knots in solid tori with solid tori surgeries.

Both Berge and Gabai recognized that knots in solid tori with solid tori surgeries must be either isotopic or "almost" isotopic to the boundary. We say that a knot K is in **bridge position** with respect to a Heegaard decomposition of a manifold if K meets each side of the Heegaard surface in a collection of arcs that are parallel to the boundary. The genus g bridge number of K, $b_g(K)$, is the minimum number of such arcs over all genus g decompositions of the manifold. Pictured is part of a handlebody together with a 1–bridge knot. Here, the knot is isotopic into the boundary except for a small arc (red). This is what we mean by "almost" isotopic to the boundary.



Wu [14] looked at surgery on 1–bridge knots in handlebodies. Based on his investigations he conjectured that H–knots in handlebodies of genus $g > 1$ should also be 1–bridge. However, we have shown [5]:

Theorem 1 (B.). *Up to homeomorphism, there are infinitely many pairs* (H, K) *where* H *is a handlebody of genus* 2*,* K *is an* H*–knot, and* K *is not* 1*–bridge in* H*.*

More recently, Ken Baker, John Luecke, and I have been able to show that many more of the knots constructed in [5] are not 1–bridge [1]. In fact, there are knots in this family with arbitrarily high bridge number:

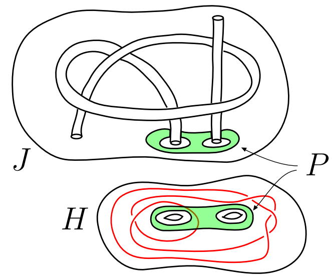
Theorem 2 (Baker-B.-Luecke). *Given* $C > 0$*, there are infinitely many pairs* (H, K) *where* H *is a handlebody of genus* 2*,* K *is an* H*–knot, and* $b_2(K) > C$*.*

The proof uses machinery of [2]. In addition to this theorem, we have generalized the construction to give examples of knots in handlebodies that have surgeries yielding the union of a Seifert fibered space over the disk with two exceptional fibers and a 1–handle.

Given such interesting knots in handlebodies, it's a natural question to ask whether they give counterexamples to important open questions.

Question 1. *How can we embed the handlebody into other manifolds to obtain interesting examples of knots in other manifolds (closed or with boundary)? For example, is it possible to obtain knots in a closed manifold with cosmetic surgeries?*

The handlebody/knot pairs constructed in Theorem 1 all contain planar surfaces which are essential in the complement of the knot. The simplest example appears at right. The handlebody has been cut along this surface, labeled P. In fact, P plays an important role in the proofs of various properties of these knots. We should then ask:



Question 2. *Given a pair (H, K) as in Theorem 1, must H contain a planar surface that is essential in the complement of K?*

The answer to this question bears on a possible classification of such knots. If the philosophy that "knots which admit simple surgeries should be simple" is accurate, then Theorem 1 shows that we need a new notion of simplicity for H–knots when the genus of H is greater than one.

Question 3. *Is there such a measure of complexity, and is the above philosophy even correct? What would a classification of H–knots look like?*

2 Bridge number in closed manifolds

A boundary reducible surgery is special because we obtain a special manifold (one which is boundary reducible). In a manifold without boundary, we can examine which surgeries yield manifolds that are special in other ways. A knot whose complement admits a metric of constant negative curvature is called **hyperbolic**, and by results of Thurston, all but finitely many surgeries on such a knot result in hyperbolic manifolds. We wish to examine those surgeries which are *not* hyperbolic, called **exceptional** surgeries.

An important class of such knots are the Berge knots – knots in S^3 which have surgeries yielding lens spaces. Jesse Johnson and I have shown that there are Berge knots with arbitrarily high genus one bridge number [6]. This is a corollary of a more general result:

Theorem 3 (B.-Johnson)**.** *Let L be a fibered link in a compact, connected, closed, orientable 3-manifold M with fiber F such that F is not a disk, annu-*

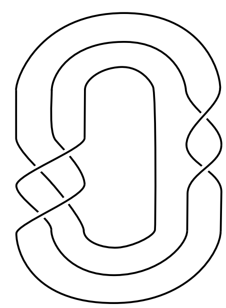
lus, or pair of pants. For any integer $D > 0$ there are infinitely many knots $K \subseteq F$ such that

$$b_g(K) > D$$

for every $0 \leq g \leq -\chi(F)$. Furthermore, we may choose these knots to be hyperbolic.

(Note that the existence of Berge knots with arbitrarily high genus one bridge number has been known for some time to both Jesse Johnson and Ken Baker.)

The proof of this theorem uses arguments from the theory of bridge splittings as well coarse geometry. Specifically, we examine the translation distance of the map on the arc and curve complex of F induced by the bundle monodromy.



Another proof of the fact that there are Berge knots with arbitrarily large genus one bridge number is given through explicit examples of twisted torus knots in work which is joint with Alex Zupan and Scott Taylor [7]. Twisted torus knots, which lie on a genus 2 splitting of S^3, are a generalization of torus knots. One such knot is pictured at right. The main theorem we prove is:

Theorem 4 (B.-Taylor-Zupan)**.** *For any $C > 0$, there is a hyperbolic twisted torus knot K and an integer p satisfying*

$$b_0(K) = p$$

and

$$C \leq b_1(K) \leq \frac{1}{2}p.$$

The construction used in this theorem applies as well to certain Dean knots, knots that have small Seifert fibered surgeries and that were studied by Dean in [8].

Although we can bound $b_1(K)$ in the above theorem, we suspect that in most cases it is possible to compute exactly. Future work will attempt to show the following (see [8] for the definition of the parameters of a twisted torus knot):

Question 4. *Is it true that, for all but finitely many s, the twisted torus knot $K_s = T(p, q, r, s)$ with $1 < r < p < q$ has*

$$b_1(K_s) = \min(r, p - r)?$$

Moriah and Sedgwick [13] have shown that some twisted torus knots have a unique minimal genus Heegaard splitting, up to isotopy. Their work uses the fact that the knots in question have genus one bridge number greater than one. On the other hand, it is known that most torus knots do not have such a unique splitting. Since little is known about the Heegaard structure of hyperbolic manifolds, answers to questions such as the following will help move the field forward:

Question 5. *Do most hyperbolic twisted torus knots have splittings of minimal genus which are unique up to isotopy? More generally, how can we describe the Heegaard structure of twisted torus knots?*

3 Topological data analysis

I am interested in applying ideas from topology and geometry to problems in data analysis. Jesse Johnson, Doug Heisterkamp, and I have been developing uses for a graph clustering algorithm due to Jesse [11]. This algorithm is based on ideas from thin position for knots and 3–manifolds, and includes several interesting features. Partly based on work done by Doug, I have developed software plugins for Cytoscape, a popular graph exploration package used by biologists, network scientists, and others, as well as scikit-learn, a Python package used by data analysts. We have applied Jesse's clustering algorithm to data sets from biology (protein interaction networks), shown that the algorithm approximates the normalized mean cut about as well as the best existing approaches, and applied the algorithm to finding optimal parameter values for feature selection.

In addition to this ongoing work, my colleague Itamar Gal and I have written software to compute the persistent homology of a point cloud using a novel method involving collapsing simplicial complexes. We have also experimented with several refinements of persistent homology.

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